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Hypergeometric Series in a Series Expansion of the Directed-Bond Percolation Probability on the Square Lattice

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The asymmetric directed-bond percolation (ADBP) problem with an asymmetry parameter k is introduced and some rigorous results are given concerning a series expansion of the percolation probability on the square lattice. It is shown that the first correction term $d_{n,1}(k)$ is expressed by Gauss' hypergeometric series with a variable k. Since the ADBP includes the ordinary directed bond percolation as a special case with $k = 1$, our results give another proof for the Baxter–Guttmann's conjecture that $d_{n,1}(1)$ is given by the Catalan number, which was recently proved by Bousquet-Mélou. Direct calculations on finite lattices are performed and combining them with the present results determines the first 14 terms of the series expansion for percolation probability of the ADBP on the square lattice. The analysis by Dlog Padé approximations suggests that the critical value depends on k, while asymmetry does not change the critical exponent β of percolation probability.

KEY WORDS: Directed bond percolation; percolation probability; asymmetry; series expansion; correction terms; hypergeometric series.

1. INTRODUCTION

The bond percolation problem on the directed square lattice, introduced by Broadbent and Hammersley,[1] has been studied by many authors[2-12] and yet there is no exact solution. Baxter and Guttmann[9] estimated the percolation probability by extrapolation from finite-lattice calculations.

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Consider a down-pointing triangular region in the square lattice with a linear size n, V_n^0 defined as

$$V_n^0 = \{(x, y) \in \mathbf{Z}^2: x + y = \text{even}, 0 \leqslant y \leqslant n-1, -y \leqslant x \leqslant y\} \tag{1.1}$$

which has $n(n-1)$ bonds. We assume that each bond is either open with probability p or closed with probability $q = 1 - p$. We say "there is an open path from (x_0, y_0) to $(x_m, y_0 + m)$" for $m \geqslant 1$ if there is a sequence (x_0, y_0), $(x_1, y_0 + 1),..., (x_m, y_0 + m)$ of sites in V_n^0 such that for each $0 \leqslant k \leqslant m-1$ the bond from $(x_k, y_0 + k)$ to $(x_{k+1}, y_0 + k + 1)$ is open. We regard two sites as connected if there is at least one open path between them. Let $P_n(q)$ be the probability that the origin $(0, 0)$ is connected to at least one site in the top row in V_n^0 and express $P_n(q)$ as a polynomial of q. Baxter and Guttmann[9] calculated $P_n(q)$ for $1 \leqslant n \leqslant 29$ and found that the difference between $P_n(q)$ and $P_{n+1}(q)$ is of order q^{n+1} and can be expressed as

$$P_n(q) - P_{n+1}(q) = q^n \sum_{l \geqslant 1} d_{n,l} d^l \tag{1.2}$$

with non-zero coefficients $\{d_{n,l}\}$, which are called the *correction terms*. This observation means the convergence of the polynomial $P_n(q)$ to a limit as a formal power series of q and this limit will define the bond percolation probability $P(q)$ for the infinite directed square lattice $V^0 = \bigcup_{n=0}^{\infty} V_n^0$. Moreover, they gave the following conjectures for the first and the second correction terms, $d_{n,1}$ and $d_{n,2}$:

$$d_{n,1} = c_n, \qquad d_{n,2} = 2c_n - c_{n+1} \tag{1.3}$$

where c_n is the Catalan number defined by[13, 14]

$$c_n = \frac{(2n)!}{n!\,(n+1)!} \tag{1.4}$$

Recently Bousquet-Mélou[15] proved these conjectures by discussing the relation between the correction terms and the numbers of compact animals[16] of appropriate types on V_n^0. Therefore the calculation by Baxter and Guttmann up to $n = 29$ gives correctly the first 31 coefficients in the series expansion of $P(q)$. Quite recently Jensen and Guttmann obtained the series $P_{39}(q)$ and determined the 41 terms of the series for $P(q)$ using the results of Bousquet-Mélou.[15] It should be noted that Baxter and Guttmann[9] and Jensen and Guttmann[12] have proposed the formulas giving higher order correction terms; however, they remain to be proved.

In a previous paper[17] we gave another proof for (1.2) and (1.3). We considered the probability $P_{n,m}$ that the origin $(0,0)$ is connected to exactly m sites in the top row in V_n^0 and studied the polynomial in the form

$$P_{n,m} = \sum_i a_{n,m,i} p^{n(n-1)-i} q^i \tag{1.5}$$

Since $P_n - P_{n+1} = \sum_{m=1}^{n} P_{n,m} q^{2n}$ by definition, it is easy to show that (1.2) is equivalent to the statement that

$$a_{n,m,i} = 0 \qquad \text{for} \quad m \geqslant 1 \quad \text{and} \quad i + m \leqslant n-1 \tag{1.6}$$

and we find that

$$\begin{aligned} d_{n,1} &= a_{n,1,n-1} \\ d_{n,2} &= a_{n,1,n} + a_{n+1,1,n} - (n^2 - 2n + 3)\, a_{n,1,n-1} \end{aligned} \tag{1.7}$$

We proved (1.6) and derived the following results:[17]

$$\begin{aligned} a_{n,m,n-m} &= b_{n+m-1,n-m} \\ a_{n,m,n-m+1} &= \{(n-1)^2 - (m-1)\}\, a_{n,m,n-m} - (m+1)\, a_{n,m+1,n-m-1} \\ &\text{for} \quad 1 \leqslant m \leqslant n \end{aligned} \tag{1.8}$$

where $b_{n,m}$ is a ballot number defined by[14]

$$b_{n,m} = \frac{n+1-m}{n+1} \binom{n+m}{m} \tag{1.9}$$

Since $c_n = b_{n,n}$, (1.3) is derived from (1.8). That is, we have generalized the statements given by Baxter and Guttmann[9] and Bousquet-Mélou[15] and showed that the Catalan numbers appearing in the correction terms are special cases of the ballot numbers in the polynomials $P_{n,m}$.

In the present paper, we report other generalizations. We consider the asymmetric directed-bond percolation problem parametrized by k, which was first studied by Domany and Kinzel.[3] It includes the usual symmetric case as $k=1$. We consider $P_{n,m}$ as a polynomial of q with coefficients which are functions of k. We show that the coefficient $a_{n,m,n-m}$ can be expressed using Gauss's hypergeometric series in the form $F(-(n-m), -n, m+1; k)$. As a corollary, we determine the first correction term $d_{n,1}$ for any $n \geqslant 1$. The ballot numbers (1.9) and the Catalan numbers (1.4) are special values at $k=1$ of the expressions using hypergeometric series.

The paper is organized as follows. In Section 2, we first define the asymmetric directed-bond percolation probability $P(q;k)$ and give a lemma which corresponds to the statement (1.6). Then we give a theorem and its corollaries. The proof of the lemma is given in Appendix A and the proof of the main theorem is given in Section 3 together with Appendix B. Applications of the present results and concluding remarks are given in Section 4.

2. ASYMMETRIC PERCOLATION PROBLEM AND RESULTS

We consider the asymmetric directed-bond percolation (ADBP) problem[3] on $V^0 = \bigcup_{n=1}^{\infty} V_n^0$, where V_n^0 is defined by (1.1). First we consider a finite lattice V_n^0, $2 \leqslant n < \infty$. Let $b^l(x, y)$ [resp. $b^r(x, y)$] denote the bond between a site (x, y) and a site $(x-1, y+1)$ [resp. $(x+1, y+1)$]. The $n(n-1)$ bonds in V_n^0 are classified into two sets, $B_n^l = \{b^l(x, y): (x, y) \in V_{n-1}^0\}$ and $B_n^r = \{b^r(x, y): (x, y) \in V_{n-1}^0\}$. In other words, B_n^l (resp. B_n^r) denotes a set of bonds connecting nearest neighbor pairs of sites in V_n^0, which go up and to the left (resp. right). Each bond in $B_n \equiv B_n^l \cup B_n^r$ is either open or closed with a given probability independently of other bonds. Here we introduce asymmetry between the probability to close bonds in B_n^l and that to close bonds in B_n^r. We assume that each bond in B_n^l (resp. B_n^r) becomes closed with probability q (resp. kq). Without loss of generality, we can assume that $0 \leqslant k \leqslant 1$. When all bonds in B_n are determined whether to be open or closed, we say that a *bond configuration* is given. For each bond configuration on B_n, open paths are defined in the same way as for the symmetric case ($k=1$) explained in Section 1 and two sites in V_n^0 are regarded as connected iff there is at least one open path between these two sites. Let $\bar{V}_n^0$ be a set of the sites in the top row in V_n^0

$$\bar{V}_n^0 = \{(x, n-1): (x, n-1) \in V_n^0\} \tag{2.1}$$

We consider all the open paths starting from the origin (0, 0) and define for $n \geqslant 2$

$$\mathcal{N}_n = \text{the number of sites in } \bar{V}_n^0 \text{ connected to } (0, 0) \tag{2.2}$$

and define the probability

$$P_{n,m}(q; k) = P(\mathcal{N}_n = m) \tag{2.3}$$

We let $P_{1,m} = \delta_{m,1}$. By definition, $P_{n,m}(q; k) = 0$ for $m < 0$ or $m > n$. Let

$$P_n(q; k) = \sum_{m=1}^{n} P_{n,m}(q; k) \tag{2.4}$$

Since $P_n(q;k)$ is nonincreasing in n for any given q and k, it has a limit

$$P(q;k) = \lim_{n \to \infty} P_n(q;k) \tag{2.5}$$

which is called the percolation probability of the ADBP. We can prove that $P(q;k)$ is nonincreasing in q for each k and the critical value $q_c(k)$ can be defined as

$$\begin{aligned} q_c(k) &= \inf\{q \in [0,1]: P(q;k) = 0\} \\ &= \sup\{q \in [0,1]: P(q;k) > 0\} \end{aligned} \tag{2.6}$$

for $0 \leqslant k \leqslant 1$. The ADBP problem is to calculate the percolation probability (2.5) and to determine $q_c(k)$. For the case $k = 0$, exact results were reported by Domany and Kinzel.[3]

The probability $P_{n,m}(q;k)$ is given as a polynomial of q as

$$P_{n,m}(q;k) = \sum_{i \geqslant 0} a_{n,m,i}(k)\, q^i \tag{2.7}$$

where $\{a_{n,m,i}(k)\}$ are polynomials of k. We can prove the following lemma.

Lemma 1. For $m \geqslant 1$,

$$a_{n,m,i}(k) = 0 \qquad \text{if} \quad i \leqslant n - m - 1 \tag{2.8}$$

We give the proof in Appendix A. Let

$$a^{(s)}_{n,m}(k) \equiv a_{n,m,(n-m)+s}(k) \tag{2.9}$$

for $1 \leqslant m \leqslant n$. Then (2.7) can be written as

$$P_{n,m}(q;k) = q^{n-m} \sum_{s \geqslant 0} a^{(s)}_{n,m}(k)\, q^s \qquad \text{for} \quad m \geqslant 1 \tag{2.10}$$

By definition, we have

$$P_n(q;k) - P_{n+1}(q;k) = \sum_{m=1}^{n} P_{n,m}(q;k)\, k^m q^{2m} \tag{2.11}$$

Combining (2.10) and (2.11) gives

$$P_n(q;k) - P_{n+1}(q;k) = q^n \sum_{l \geqslant 1} d_{n,l}(k)\, q^l \tag{2.12}$$

with

$$d_{n,l}(k) = \sum_{m=1}^{l} k^m a_{n,m}^{(l-m)}(k) \tag{2.13}$$

In particular the first correction term is

$$d_{n,1}(k) = k a_{n,1}^{(0)}(k) \tag{2.14}$$

Now we state the main theorem in the present paper.

Theorem 2. For $1 \leqslant m \leqslant n$,

$$a_{n,m}^{(0)}(k) = \binom{n-1}{m-1} F(-(n-m), -n, m+1; k) \tag{2.15}$$

Here $F(\alpha, \beta, \gamma; z)$ is Gauss' hypergeometric series, which is defined as

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \tag{2.16}$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n \geqslant 1$ and $(\alpha)_0 = 1$. Since Gauss' summation theorem

$$F(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \tag{2.17}$$

is valid if $\Re\gamma > 0$, $\Re(\gamma-\alpha-\beta) > 0$, we can see that

$$a_{n,m}^{(0)}(1) = \frac{2m}{n+m} \binom{2n-1}{n+m-1} \tag{2.18}$$

and we have the following corollary, which was given in Theorem 2 of ref. 17.

Corollary 3. In particular, for $k = 1$

$$a_{n,m}^{(0)}(1) = b_{n+m-1,n-m} \tag{2.19}$$

where $b_{n,m}$ is the ballot number defined by (1.9).

We have given a simple relation (2.14), and thus the next corollary is derived from Theorem 2.

Corollary 4. Let $d_{n,1}(k)$ be the first correction term for the ADBP with a parameter k. Then

$$d_{n,1}(k) = kF(-(n-1), -n, 2; k) \tag{2.20}$$

for $n \geqslant 1$. In particular, for the symmetric case,

$$d_{n,1}(1) = c_n \tag{2.21}$$

where c_n is the Catalan number defined by (1.4).

3. PROOF OF THEOREM 2

We can find that $a^{(0)}_{n,m}(k)$ with $1 \leqslant m \leqslant n$ is given by the solution of the difference equation

$$\alpha_{n+1,m} = k\alpha_{n,m+1} + (1+k)\alpha_{n,m} + \alpha_{n,m-1} \qquad \text{for} \quad n \geqslant 1 \tag{3.1}$$

with the boundary conditions

$$\alpha_{1,m} = \delta_{m,1} \tag{3.2}$$

and

$$\alpha_{n,m} = 0 \qquad \text{if} \quad m \leqslant 0 \quad \text{or} \quad m \geqslant n+1 \tag{3.3}$$

The derivation is given in Appendix B.

It should be noticed that (3.1)–(3.3) give

$$\alpha_{n,n} = 1 \qquad \forall n \geqslant 1 \tag{3.4}$$

We introduce the generating function as

$$\Phi(x, y; k) = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \alpha_{n,m} x^n y^m \tag{3.5}$$

It is easy to obtain

$$\Phi(x, y; k) = \frac{xy(ka(x; k) - y)}{x(y+1)(y+k) - y} \tag{3.6}$$

with

$$a(x; k) = \sum_{n-1}^{\infty} \alpha_{n,1} x^n \tag{3.7}$$

The function $a(x;k)$ can be determined by the condition (3.4) as explained below. Let

$$\Phi(x, y; k) = \sum_{m=1}^{\infty} \phi_m(x; k)\, y^m \tag{3.8}$$

Then by (3.3) and (3.4)

$$\phi_m(x; k) = x^m + \mathcal{O}(x^{m+1}) \qquad \text{for} \quad m \geqslant 1 \tag{3.9}$$

The solution (3.6) gives

$$\phi_1(x; k) = a(x; k) \tag{3.10}$$

$$\phi_2(x; k) = -\frac{1}{k} - \left(1 + \frac{1}{k} - \frac{1}{kx}\right) a(x; k) \tag{3.11}$$

$$\phi_3(x; k) = \frac{1}{k}\left(1 + \frac{1}{k} - \frac{1}{kx}\right) + \frac{1}{k^2}\left[(1 + k + k^2) - 2(1 + k)\frac{1}{x} + \frac{1}{x^2}\right] a(x; k) \tag{3.12}$$

By (3.10) and (3.9) with $m = 1$, we have

$$a(x; k) = x + \mathcal{O}(x^2) \tag{3.13}$$

Assume that we can solve the equation $a = a(x; k)$ and obtain x as a function of a and k. We consider the Laurent expansion of $1/x$ with respect to a as

$$\frac{1}{x} = \sum_{l=-\infty}^{\infty} \delta_l(k)\, a^l \tag{3.14}$$

and rewrite the RHS of Eqs. (3.10)–(3.12) as functions of a and k. Following (3.13), the condition (3.9) becomes

$$\phi_m(a; k) = a^m + \mathcal{O}(a^{m+1}) \qquad \text{for} \quad m \geqslant 1 \tag{3.15}$$

By the condition (3.15) with $m = 1, 2, 3$, we have $\delta_l(k) = 0$ for $l \leqslant -2$, $\delta_{-1}(k) = 1$, $\delta_0(k) = 1 + k$, $\delta_1(k) = k$, and $\delta_2(k) = \delta_3(k) = 0$. This result suggests that $1/x = (1 + ka)(1 + a)/a$, or equivalently

$$x = \frac{a}{(1 + ka)(1 + a)} \tag{3.16}$$

If we assume (3.16), we can write (3.6) as

$$\Phi(x, y; k) = \frac{ay}{1-ay}$$

$$= \sum_{m=1}^{\infty} a^m y^m \tag{3.17}$$

which satisfies the condition (3.15). Then (3.16) is correct. The relation (3.16) gives a quadratic equation for a and is solved as

$$a = \{1-(1+k)\,x \pm [1-2(1+k)\,x+(1-k)^2\,x^2]^{1/2}\}/(2kx)$$

It should be identified with the equation $a = a(x; k)$ with (3.13). we conclude

$$a(x; k) = \frac{1}{2kx}\{1-(1+k)\,x-[1-2(1+k)\,x+(1-k)^2\,x^2]^{1/2}\} \tag{3.18}$$

It is easy to confirm that

$$a(x; k) = \sum_{n=1}^{\infty} F(-(n-1), -n, 2; k)\,x^n \tag{3.19}$$

where $F(\alpha, \beta, \gamma; z)$ is Gauss' hypergeometric series defined as in (2.16). Then (3.7) implies that

$$\alpha_{n,1} = F(-(n-1), -n, 2; k) \qquad \text{for} \quad n \geqslant 1 \tag{3.20}$$

Equation (3.1) with (3.3) gives

$$k\alpha_{n,2} = \alpha_{n+1,1} - (1+k)\,\alpha_{n,1} \tag{3.21}$$

and

$$k\alpha_{n,m+1} = \alpha_{n+1,m} - (1+k)\,\alpha_{n,m} - \alpha_{n,m-1} \qquad \text{for} \quad m \geqslant 2 \tag{3.22}$$

Combining (3.20) and (3.21) gives

$$\alpha_{n,2} = (n-1)\,F(-(n-2), -n, 3; k) \tag{3.23}$$

Therefore we have proved that (2.15) is valid for $m = 1$ and 2. For $m \geqslant 3$, (2.15) can be proved by induction using (3.22).

Remark. Bousquet-Mélou[15] showed that $d_{n,1}(1)$ is the number of *staircase animals*[18] of directed height n. The fact that (B.3) holds and $a^{(0)}_{n+1,m,1}(k)$ is completely determined by $a^{(0)}_{n,m',1}(k)$ as in (B.4) means that $a^{(0)}_{n,m}(1)$ given by (2.18) is the number of the compact bond animals which have exactly m sites at the directed height n. These animals are closely related to *convex polygons*, which are defined as self-avoiding polygons whose number of steps equals the perimeter of their minimal bounding rectangle.[19–21]

4. APPLICATIONS AND REMARKS

We performed direct calculations of $P_{n,m}(q;k)$ in order to check (2.15) numerically and demonstrate how to extend the series of percolation probability using the correction term formula (2.12). In comparison with the calculations by Baxter and Guttmann[9] and Jensen and Guttmann[12] of $P_n(q)$ for the ordinary directed-bond percolation, much more memory storages are needed for the present calculation. The reasons are as follows: (i) We consider the probabilities $P_{n,m}(q;k)$ having m "particles" at the nth level for every $m \in \{1, 2,..., n\}$ instead of only considering their sum $P_n(q;k) = \sum_{m=1}^{n} P_{n,m}(q;k)$. (ii) For the ADBP, the coefficients $a_{n,m,i}(k)$ in the series $P_{n,m}(q;k)$ are not numbers, but polynomials of k, which have at most n terms. We have obtained $P_{n,m}(q;k)$ up to $n = 12$ using *Mathematica* and the result directly determines the first 12 terms of the series for the percolation probability. Compared to the 39-term series for the symmetric case by Jensen and Guttmann[12], our series for the ADBP is relatively short. In order to obtain longer series, we have to make some special programs as reported in the previous papers for the symmetric case.[9,12] Moreover, in order to calculate the polynomials $a_{n,m,i}(k)$ correctly, new algorithms are required; e.g., first we calculate values of $a_{n,m,i}(k)$ for sufficiently many but finite number of values of k and then determine the coefficients of power series of k by solving simultaneous linear equations. In the present paper, however, we report the results only using the data up to $n = 12$ and leave higher order calculation as a future problem, since the main purpose of the paper is to show that the hypergeometric series appears in series expansions for the ADBP and that it leads as a special case to the appearance of the Catalan numbers in the correction terms for the symmetric case.

We determined the coefficients $a_{n,m}^{(s)}(k)$ of (2.10) up to $n = 12$ and found that all results for $s = 0$ satisfy (2.15). Let

$$P_n(q;k) = \sum_{i \geq 0} a_{n,i}(k)\, q^i \tag{4.1}$$

Lemma 1 implies that $P_n(q;k)$ converges to a limit $P_\infty(q;k)$ in $n \to \infty$ as a formal power series of q. Let

$$P_\infty(q;k) = \sum_{n \geq 0} a_n(k)\, q^n \tag{4.2}$$

Then Lemma 1 also states that $a_n(k) = a_{n,n}(k)\ \forall n \geqslant 1$ and thus our direct calculation determines $a_i(k)$ for $i = 1$–12. Since we have $\{a_{12,m}^{(s)}(k)\}$, the correction terms $d_{n,l}(k)$ can be obtained for $n = 12$ by (2.13). In particular, we see that

$$\begin{aligned} d_{12,1}(k) &= ka_{12,1}^{(0)}(k) \\ &= k + 66k^2 + 1210k^3 + 9075k^4 + 32670k^5 + 60984k^6 \\ &\quad + 60984k^7 + 32670k^8 + 9075k^9 + 1210k^{10} + 66k^{11} + k^{12} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} d_{12,2}(k) &= ka_{12,1}^{(1)}(k) + k^2 a_{12,2}^{(0)}(k) \\ &= -11k^2 - 440k^3 - 5445k^4 - 29040k^5 - 76230k^6 - 104544k^7 \\ &\quad - 76230k^8 - 29040k^9 - 5445k^{10} - 440k^{11} - 11k^{12} \end{aligned} \tag{4.4}$$

The formula (2.12) and definition (4.1) give that

$$\begin{aligned} a_{13}(k) &= a_{13,13}(k) \\ &= a_{12,13}(k) - d_{12,1}(k) \\ &= -k - 11k^2 - 91k^3 - 439k^4 - 1276k^5 - 2185k^6 \\ &\quad - 2185k^7 - 1276k^8 - 439k^9 - 91k^{10} - 11k^{11} - k^{12} \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} a_{13,14}(k) &= a_{12,14}(k) - d_{12,2}(k) \\ &= 66k^2 + 1605k^3 + 15116k^4 + 68666k^5 + 165411k^6 + 220746k^7 \\ &\quad + 165411k^8 + 68666k^9 + 15116k^{10} + 1605k^{11} + 66k^{12} \end{aligned} \tag{4.6}$$

Now we use Corollary 4. It gives

$$\begin{aligned} d_{13,1}(k) &= kF(-12, -13, 2, k) \\ &= k + 78k^2 + 1716k^3 + 15730k^4 + 70785k^5 + 169884k^6 \\ &\quad + 226512k^7 + 169884k^8 + 70785k^9 + 15730k^{10} \\ &\quad + 1716k^{11} + 78k^{12} + k^{13} \end{aligned} \tag{4.7}$$

and we have

$$
\begin{aligned}
a_{14}(k) &= a_{14,\,14}(k) \\
&= a_{13,\,14}(k) - d_{13,\,1}(k) \\
&= -k - 12k^2 - 111k^3 - 614k^4 - 2119k^5 - 4473k^6 - 5766k^7 \\
&\quad - 4473k^8 - 2119k^9 - 614k^{10} - 111k^{11} - 12k^{12} - k^{13}
\end{aligned} \tag{4.8}
$$

Thus we have extended the series (4.2) by two terms. The results are summarized in Table I.

Using the obtained series, we estimated the critical value $q_c(k)$ and the critical exponent β of percolation probability for several k by the Dlog Padé approximations.[9, 12] The results are shown in Table II. It is shown that $q_c(k)$ decreases monotonically in k, while β does not depend on k for $0 < k < 1$. The estimated values for β are very close to the value $\beta = 0.27643 \pm 0.00010$ by Jensen and Guttmann using a long series for the symmetric case $k = 1$.[12] The present analysis strongly suggests that asymmetry does not change the critical exponent β. It should be noted that Martins *et al.*[22] reported that they estimated the critical exponent by

Table I. The Numbers $c_{n,r}$ in the Polynomials $a_n(k) = \sum_{n \geq 0} c_{n,r} k^r$ for the Coefficients in the Series Expansion $P(q; k) = \sum_{n \geq 0} a_n(k)\, q^n$ for the ADBP on the Square Lattice

	$c_{n,r}$													
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1													
1	0													
2	0	-1												
3	0	-1	-1											
4	0	-1	-2	-1										
5	0	-1	-3	-3	-1									
6	0	-1	-4	-7	-4	-1								
7	0	-1	-5	-13	-13	-5	-1							
8	0	-1	-6	-21	-32	-21	-6	-1						
9	0	-1	-7	-31	-66	-66	-31	-7	-1					
10	0	-1	-8	-43	-119	-169	-119	-43	-8	-1				
11	0	-1	-9	-57	-196	-369	-369	-196	-57	-9	-1			
12	0	-1	-10	-73	-301	-718	-959	-718	-301	-73	-10	-1		
13	0	-1	-11	-91	-439	-1276	-2185	-2185	-1276	-439	-91	-11	-1	
14	0	-1	-12	-111	-614	-2119	-4473	-5766	-4473	-2119	-614	-111	-12	-1

Table II. Critical Values $q_c(k)$ and Critical Exponents β for Several k Estimated by the Dlog Padé Approximations

	q_c			β		
k	[7/6]	[6/7]	[6/6]	[7/6]	[6/7]	[6/6]
0.01	0.9687	0.9687	0.9687	0.2777	0.2776	0.2776
0.1	0.7814	0.7816	0.7817	0.2760	0.2778	0.2787
0.2	0.6644	0.6650	0.6647	0.2764	0.2777	0.2791
0.3	0.5868	0.5869	0.5872	0.2765	0.2776	0.2796
0.4	0.5301	0.5302	0.5308	0.2766	0.2774	0.2710
0.5	0.4861	0.4861	0.4858	0.2769	0.2772	0.2818
0.6	0.4506	0.4506	0.4505	0.2772	0.2772	0.2746
0.7	0.4210	0.4210	0.4210	0.2774	0.2777	0.2771
0.8	0.3960	0.3963	0.3960	0.2776	0.2782	0.2776
0.9	0.3743	0.3739	0.3743	0.2777	0.2686	0.2777
1.0	0.3554	0.3555	0.3554	0.2777	0.2723	0.2778

Table III. Correction Terms $d_{n,l}(k)$ for Small n and l

	$d_{n,l}(k)$		
n	$l=1$	$l=2$	$l=3$
1	k	0	0
2	$k+k^2$	$-k^2$	$-k^2-k^3$
3	$k+3k^2+k^3$	$-2k^2-2k^3$	$-3k^2-6k^3-3k^4$
4	$k+6k^2+6k^3+k^4$	$-3k^2-8k^3-3k^4$	$-6k^2-22k^3-22k^4-6k^5$
5	$k+10k^2+20k^3+10k^4+k^5$	$-4k^2-10k^3-10k^4-4k^5$	$-10k^2-66k^3-100k^4-66k^5-10k^6$

	$d_{n,l}(k)$	
n	$l=4$	$l=5$
1	0	0
2	k^3	0
3	$5k^3+5k^4$	$3k^3+4k^4+3k^5$
4	$13k^3+28k^4+13k^5$	$13k^3+38k^4+38k^5+13k^6$
5	$27k^3+98k^4+98k^5+27k^6$	$35k^3+165k^4+262k^5+165k^6+35k^7$

Monte Carlo simulation as $\beta = 0.5 \pm 0.02$ for the asymmetric Domany–Kinzel cellular automata. Although we could not find the details of their estimation, our result contradicts their value.

Finally we make one more remark. For the symmetric case $k=1$, Bousquet-Mélou[15] proved that the second correction term $d_{n,2}$ is also expressed by the Catalan numbers as in (1.3). In the previous paper,[17] we proved that not only the $a^{(0)}_{n,m}$ but also the $a^{(1)}_{n,m}$ are expressed using ballot numbers as (1.8). Table III shows correction terms $d_{n,l}(k)$ for small n and l obtained by direct calculations. We find that the following relation holds for any n up to $n=12$:

$$d_{n,2}(k) = (1+k)\, d_{n,1}(k) - d_{n+1,1}(k) \tag{4.9}$$

Although we have not yet proved this formula, it is plausible, since it becomes the second equation of (1.3) in the limit $k=1$. The intrepid reader could attempt to guess the formulas for subsequent correction terms using Table III. It is a challenging problem to prove them and show that, for $0<k<1$, $d_{n,l}(k)$ with $l \geqslant 2$ and $a^{(l)}_{n,m}(k)$ with $l \geqslant 1$ are also expressed by using Gauss' hypergeometric series.

APPENDIX A. PROOF OF LEMMA 1

We consider the ADBP on V^0. For a given bond configuration on B_n, $n \geqslant 2$, define a set of integers

$$A^0_n = \{x \in \mathbf{Z}\colon (x, n-1) \in \bar{V}^0_n \text{ and } (x, n-1) \text{ is corrected to}(0,0)\} \tag{A.1}$$

For the convenience, we define $\bar{V}^0_1 = A^0_1 = \{0\}$. If we write the number of elements in a set S as $|S|$, then $|A^0_1| = 1$ and $|A^0_n| = \mathcal{N}_n$ for $n \geqslant 2$, where $\mathcal{N}_n$ is defined by (2.2). Let $\mathbf{Z}_e = \{..., -4, -2, 0, 2, 4,...\}$ and $\mathbf{Z}_o = \{..., -3, -1, 1, 3,...\}$. By definition, $A^0_n \subset \mathbf{Z}_e$ if n = odd and $A^0_n \subset \mathbf{Z}_o$ if n = even. If $x, y \in A^0_n$ and $|x-y| = 2$, we say that x and y are adjacent. We say also that a sequence of adjacent sites in A^0_n, $\{l+1, l+3,..., r-3, r-1\}$, makes a *cluster* if $l-1 \notin A^0_n$ and $r+1 \notin A^0_n$. It is noted that a single site $\{x\}$ can make a cluster when $x \in A^0_n$ but $x-2 \notin A^0_n$ and $x+2 \notin A^0_n$. We can write $A^0_n = \bigcup_{k=1}^{c} A^0_{n,k}$ where $A^0_{n,k} = \{l_k+1, l_k+3,..., r_k-3, r_k-1\}$, $k = 1, 2,..., c$, are clusters of which A^0_n consists. The number of clusters characterizes A^0_n and is denoted by $c(A^0_n)$.[17]

Let Y_n be the collection of all subsets of $\{-n+1, -n+3,..., n-3, n-1\}$, $n \geqslant 1$. For $B \in Y_n$, we have

$$P(A^0_n = B) = \sum_{i \geqslant 0} a_{n,B,i}(k)\, q^i \tag{A.2}$$

where $\{a_{n,B,i}(k)\}$ are polynomials of k. Define the indicator function $1_{\{\omega\}}$ as $1_{\{\omega\}}=1$ if ω is true, $=0$ otherwise. Since $P_{n,m}(q;k)=\sum_{B\in Y_n} 1_{\{|B|=m\}} P(A_n^0=B)$, Lemma 1 follows once we prove the next lemma.

Lemma A.1. For $B\in Y_n$ with $|B|\geqslant 1$,

$$a_{n,B,i}(k)=0 \qquad \text{if} \quad i\leqslant n-|B|-1 \tag{A.3}$$

In this Appendix, we will prove Lemma A.1 by induction with respect to n.

When $n=1$, $P(A_1^0=B)=1_{\{B=\{0\}\}}$ by definition, for which (A.3) is correct. We assume that (A.3) holds for $n=N\geqslant 1$. In other words

$$P(A_N^0=B)=a_{N,B}^{(0)}(k)\,q^{N-m'}+\mathcal{O}(q^{N-m'+1}) \tag{A.4}$$

for $B\in Y_n$ with $m'=|B|$, where $a_{N,B}^{(0)}(k)=a_{N,B,N-m'}(k)$. We use the following formula.

$$P(A_{N+1}^0=C)=\sum_{B\in Y_N} q_N(C,B)\,P(A_N^0=B) \tag{A.5}$$

when $C\in Y_{N+1}$ and $q_N(C,B)$ is the conditional probability

$$q_N(C,B)=P(A_{N+1}^0=C|A_N^0=B) \tag{A.6}$$

Now we study $q_N(C,B)$. Assume that $|B|=m'\geqslant 1$, $c(B)=c'$, $|C|=m$, and $B=\bigcup_{k=1}^{c'} B_k$, where $B_k=\{l_k+1, l_k+3,\ldots, r_k-3, r_k-1\}$, $k=1,2,\ldots,c'$, are the clusters of B. Let

$$\hat{B}_k=\{l_k+2, l_k+4,\ldots, r_k-4, r_k-2\}$$

$$\check{B}_k=\{l_k, l_k+2,\ldots, r_k-2, r_k\}$$

and

$$\hat{B}_k=\bigcup_{k=1}^{c'} \hat{B}_k, \qquad \check{B}=\bigcup_{k=1}^{c'} \check{B}_k$$

We classify $q_N(C,B)$ according to the value $\Delta m\equiv m-m'$. By the construction of the ADBP, we have the following results.

(1) $\Delta m>c'$: $q_N(C,B)=0$.

(2) $\Delta m = c'$:

$$q_N(C, B) = \begin{cases} 1 + \mathcal{O}(q) & \text{if} \quad C = \check{B} \\ 0 & \text{otherwise} \end{cases}$$

(3) $0 \leqslant \Delta m < c'$: if C can be written as

$$C = \check{B} \Big\backslash \left(\bigcup_{s=1}^{a} \{l_{i_s}\} \cup \bigcup_{s=1}^{b} \{r_{j_s}\} \right)$$

with $1 \leqslant i_1 < i_2 < \cdots < i_a \leqslant c'$

and $1 \leqslant j_1 < j_2 < \cdots < j_b \leqslant c'$

where $i_s \neq j_l$ for any pair (s, l) and $a + b = c' - \Delta m$ (A.7)

then $q_N(C,B) = k^b q^{c' - \Delta m} + \mathcal{O}(q^{c' - \Delta m + 1})$, otherwise $q_N(C,B) = \mathcal{O}(q^{c' - \Delta m + 1})$.

(4) $-c' < \Delta m \leqslant -1$: if C can be written as

$$C = \hat{B} \cup \left(\bigcup_{s=1}^{a} \{l_{i_s}\} \cup \bigcup_{s=1}^{b} \{r_{j_s}\} \right)$$

with $1 \leqslant i_1 < i_2 < \cdots < i_a \leqslant c'$

and $1 \leqslant j_1 < j_2 < \cdots < j_b \leqslant c'$

where $i_s \neq j_l$ for any pair (s, l) and $a + b = c' + \Delta m$ (A.8)

then $q_N(C,B) = k^{c'-b} q^{c' - \Delta m} + \mathcal{O}(q^{c' - \Delta m + 1})$, otherwise $q_N(C,B) = \mathcal{O}(q^{c' - \Delta m + 1})$.

(5) $\Delta m = -c'$:

$$q_N(C, B) = \begin{cases} k^{c'} q^{2c'} + \mathcal{O}(q^{2c'+1}) & \text{if} \quad C = \hat{B} \\ \mathcal{O}(q^{2c'+1}) & \text{otherwise} \end{cases}$$

(6) $\Delta m < -c'$:

$$q_N(C, B) = \begin{cases} k^{-\Delta m} q^{-2\Delta m} + \mathcal{O}(q^{-2\Delta m + 1}) & \text{if} \quad C \subset \hat{B} \\ \mathcal{O}(q^{-2\Delta m + 1}) & \text{otherwise} \end{cases}$$

Under the assumption (A.4), we find, for a given $B \in Y_n$ with $m' = |B|$ and $c(B) = c'$,

$$q_N(C, B)\, P(A_N^0 = B) = Q(C, B; k)\, a_{N,B}^{(0)}(k)\, q^{N - m + c'} + \mathcal{O}(q^{N - m + c' + 1}) \quad \text{(A.9)}$$

with

$$Q(C,B;k)=\begin{cases} 1 & \text{if } C=\check{B} \\ k^b & \text{if } 0\leqslant \Delta m<c' \\ & \quad \text{and } C \text{ is written in the form (A.7)} \\ k^{c'-b} & \text{if } -c'<\Delta m\leqslant -1 \\ & \quad \text{and } C \text{ is written in the form (A.8)} \\ k^{c'} & \text{if } C=B \\ 0 & \text{otherwise} \end{cases} \tag{A.10}$$

where $\Delta m=m-m'$ with $m=|C|$. The degree $N-m+c'$ is minimized when $c'=1$.

Assume that $c'=1$. Then B is written in the form $B=\{l+1, l+3,..., r-1\}$ and we conclude that

$$q_N(C,B)\,P(A_N^0=B)=\tilde{Q}(C,B;k)\,a_{N,B}^{(0)}(k)\,q^{(N+1)-m}+\mathcal{O}(q^{(N+1)-m+1}) \tag{A.11}$$

with

$$\tilde{Q}(C,B;k)=\begin{cases} 1 & \text{if } C=\{l,l+2,...,r\} \text{ or } \{l+2,l+4,...,r\} \\ k & \text{if } C=\{l,l+2,...,r-2\} \text{ or } \{l+2,l+4,...,r-2\} \\ 0 & \text{otherwise} \end{cases} \tag{A.12}$$

Since (A.5) holds, this completes the proof of Lemma A.1.

APPENDIX B. DERIVATION OF THE DIFFERENCE EQUATION

Define

$$P_{n,m,c}(q;k)=P(|A_n^0|=m,\ c(A_n^0)=c) \tag{B.1}$$

Since $P_{n,m,c}(q;k)=\sum_{B\in Y_n} 1_{\{|B|=m,\,c(B)=c\}}\,P(A_n^0=B)$, Lemma A.1 gives

$$P_{n,m,c}(q;k)=q^{n-m}\sum_{s\geqslant 0} a_{n,m,c}^{(s)}(k)\,q^s \tag{B.2}$$

if $m\geqslant 1$, where the coefficients $\{a_{n,m,c}^{(s)}(k)\}$ are expressed by $\{a_{n,B,i}(k)\}$ with $|B|=m$ and $c(B)=c$. In the proof of Lemma A.1 given in Appendix A, we showed that, for $B=\{l+1,l+3,...,r-1\}$, $q_n(C,B)\,P(A_n^0=B)$ has a term in the degree $q^{(n+1)-m}$ if and only if $c'=c(B)=1$ and $C\in\mathscr{A}(B)\equiv\{\hat{B}, \hat{B}\cup\{l\},$

$\hat{B} \cup \{r\}, \breve{B}\}$, otherwise it has only terms in higher order of q. Since $c(C)=1$ for $C \in \mathscr{A}(B)$, $a^{(0)}_{n,m,c}(k) \neq 0$ iff $c=1$ for $m \geqslant 1$. By the definitions (2.3) and (B.1), $\sum_{c \geqslant 1} P_{n,m,c}(q;k) = P_{n,m}(q;k)$ and $P_{1,m,c}(q;k) = 1_{\{m=1,c=1\}}$. Thus we have an important result,

$$a^{(0)}_{n,m}(k) = a^{(0)}_{n,m,1}(k) \qquad \text{for} \quad m \geqslant 1 \tag{B.3}$$

Since $a^{(0)}_{n,m,1}(k) = \sum_{B \in Y_n} 1_{\{|B|=m, c(B)=1\}} a^{(0)}_{n,B}(k)$, the formula (A.5) and the result (A.11) with (A.12) in Appendix A give that

$$\begin{aligned}
&a^{(0)}_{n+1,1,1}(k) = ka^{(0)}_{n,2,1}(k) + (1+k)\, a^{(0)}_{n,1,1}(k) \\
&a^{(0)}_{n+1,m,1}(k) = ka^{(0)}_{n,m+1,1}(k) + (1+k)\, a^{(0)}_{n,m,1}(k) + a^{(0)}_{n,m-1,1}(k) \\
&\qquad\qquad \text{for} \quad 2 \leqslant m \leqslant n-1 \\
&a^{(0)}_{n+1,n,1}(k) = (1+k)\, a^{(0)}_{n,n,1}(k) + a^{(0)}_{n,n-1,1}(k) \\
&a^{(0)}_{n+1,n+1,1}(k) = a^{(0)}_{n,n,1}(k)
\end{aligned} \tag{B.4}$$

and

$$a^{(0)}_{1,m,1}(k) = \delta_{m,1} \tag{B.5}$$

Combining these equations with (B.3) gives the difference equation (3.1) with (3.2) and (3.3).

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